10/18

Ex Find the points on a sphere $x^2+y^2+z^2=4$ closest to and farthest from $(3,-2,1)$

> need to rewrite the problem to $\begin{cases} \text{optimize} : \text{distance} \\ \text{subject to} : \text{sphere} \end{cases}$

in other words want to: $\begin{cases} \text{optimize} : \text{distance}((x,y,z),(3,-2,1)) \\ \text{subject to} : x^2+y^2+z^2=4 \end{cases}$

* want to get rid of $\sqrt{\ }$ (in dist formula) $\Rightarrow$ so equivalently we want to $\begin{cases} \text{optimize } \text{dist}^2 \\ \text{subject to sphere} \end{cases}$

$$\begin{cases} \text{optimize } (x-3)^2+(y+2)^2+(z-1)^2 \\ \text{sub to} : x^2+y^2+z^2=4 \end{cases} \Rightarrow \begin{cases} \overbrace{(x^2-6x+9)+(y^2+4y+4)+(z^2-2z+1)}^{\text{not necessary, but makes easier}} \\ = \quad '' \end{cases}$$

$$\begin{cases} (x^2+y^2+z^2)+9+4+1+(-6x+4y-2z) \\ \quad '' \end{cases}$$

need to do this to be in form for Lagrange multiplier

$$\Rightarrow \begin{cases} (4)+14+(-6x+4y-2z) \\ x^2+y^2+z^2-4=0 \end{cases}$$

• just moved 4 over

$$\begin{cases} f(x,y,z) = (18+(-6x+4y-2z)) \\ \text{sub to} : g(x,y,z)=0 \text{ for } g(x,y,z) = x^2+y^2+z^2-4 \end{cases}$$

• Now w/ $F(x,y,z,\lambda) = f(x,y,z) - \lambda g(x,y,z)$
$= 18-6x+4y-2z - \lambda(x^2+y^2+z^2-4)$

we solve $\nabla f = \vec{0}$

$\nabla f \langle -6-2\lambda x,\ 4-2\lambda y,\ -2-2\lambda z,\ -(x^2+y^2+z^2-4) \rangle$

$$\therefore \nabla f = \vec{0} \text{ iff } \begin{cases} -6-2\lambda x=0 \\ 4-2\lambda y=0 \\ -2-2\lambda z=0 \\ -(x^2+y^2+z^2-4)=0 \end{cases} \text{ iff } \begin{cases} \lambda x = -3 & (1) \\ \lambda y = 2 & (2) \\ \lambda z = -1 & (3) \\ x^2+y^2+z^2=4 & (4) \end{cases}$$

* notice 1st λ can't equal $\emptyset$ by eqn (1)

w/ eqn (4) : $x^2 + y^2 + z^2 = 4$ (could've also divided eqns 1-3 by λ & plugged into eqn (4) to solve for λ)

multiply both sides by λ^2

$\Rightarrow \lambda^2(x^2 + y^2 + z^2) = \lambda^2 4$

$\downarrow$

$(\lambda x)^2 + (\lambda y)^2 + (\lambda z)^2 = 4\lambda^2$ > now plug in (1), (2), (3)

$(-3)^2 + (2)^2 + (-1)^2 = 4\lambda^2$

$9 + 4 + 1 = 14 = 4\lambda^2$

$\therefore \lambda = \pm\sqrt{\frac{7}{2}}$ (means we will have 2 points)

> if $\lambda = \sqrt{\frac{7}{2}}$ then solving (1), (2), (3) for x, y, z will yield point $(-3\sqrt{\frac{2}{7}}, 2\sqrt{\frac{2}{7}}, -\sqrt{\frac{2}{7}}) = A$

(divided by λ in each eqn $\therefore \sqrt{\frac{7}{2}} \Rightarrow \sqrt{\frac{2}{7}}$)

compute $f(A) = 18 - 6(-3\sqrt{\frac{2}{7}}) + 4(2\sqrt{\frac{2}{7}}) - 2(-\sqrt{\frac{2}{7}})$

$= 18 + 18\sqrt{\frac{2}{7}} + 8\sqrt{\frac{2}{7}} + 2\sqrt{\frac{2}{7}}$

$= 18 + 28\sqrt{2/7}$

if $\lambda = -\sqrt{\frac{7}{2}}$ then solving eqns 1-3 for (x, y, z) yields $(3\sqrt{2/7}, -2\sqrt{2/7}, \sqrt{2/7}) = B$

$f(B) = 18 - 6(3\sqrt{\frac{2}{7}}) + 4(-2\sqrt{\frac{2}{7}}) - 2(\sqrt{\frac{2}{7}})$

$= 18 - 18\sqrt{\frac{2}{7}} - 8\sqrt{\frac{2}{7}} - 2\sqrt{2/7}$

$= 18 - 28\sqrt{2/7}$

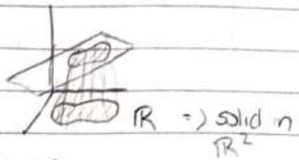
$f(A) > f(B)$ $\therefore$ *showing (noting $f(A) > f(B)$), A is* the furthest point from $(3, -2, 1)$ and B is closest to $(3, -2, 1)$ by Lagrange multipliers

Exersize: find max volume of a (rectilinear) box w/ NO lid & surface area 12

15.1 : Double Integrals

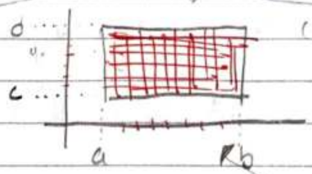
Goal : to integrate fxn of 2 variables
↳ What should an integral of 2 variables mean here ?

> in Calc 1 , it computed the net area under graph of 'f'
> in Calc 3 :
- should represent the 'net volume under the graph of 'f' and above R'



> today we'll work w/ simplest possible regions which are rectangles

$R = [a, b] \times [c, d]$ (not cross product) (rectangle whose x ranges from a to b and y component ranges from interval c to d)
$= \{(x,y) : x \in [a, b], y \in [c, d]\}$



> in calc 1 , to compute the area (definite integral) $\int_a^b f(x)\,dx$, we chunked the interval [a, b] and we appoximate area via 'left end pts' computation , adding rectangle areas w/ height f(endpts)

> in calc 3 : $\iint_R f(x,y)\,dA$ is approximated by by 'chunking' R and then using $\underbrace{f(\text{lower left endp})}_{\text{just included as example}}$ for height of the rectangular box

> now limit the approximations (don't want to, very hard to do)

> new way to solve includes fixing $y_0 \Rightarrow f(x, y_0)$ to integrate on x (better defined on next page)

Fubini's Theorem

> if $f(x,y)$ is cts. on $R = [a,b]\times[c,d]$, then

$$\int_{y=c}^{d}\left(\int_{x=a}^{b} f(x,y)\,dx\right)dy = \iint_R f(x,y)\,dA = \int_{x=a}^{b}\left(\int_{y=c}^{d} f(x,y)\,dy\right)dx$$

* all this is saying that you could've fixed x_0 1st instead of y_0

* this is hard ∴ the proof of this result is beyond scope of this course *

Ex compute $\iint_R x\sec^2(y)\,dA$ where $R = [1,3]\times[0,\frac{\pi}{4}]$

Sol 1: $\iint_R x\sec^2(y)\,dA = \int_{y=0}^{\pi/4}\int_{x=1}^{3} x\sec^2(y)\,dx\,dy$

inner int: $\int_1^3 x\sec^2(y)\,dx = \sec^2(y)\int_1^3 x\,dx = \sec^2(y)\left.\frac{x^2}{2}\right|_1^3$

$= \sec^2(y)\,\frac{1}{2}(9-1) = \sec^2(y)(4)$

$\therefore \iint_R \sec^2(y)\,dA = \int_{y=0}^{\pi/4} 4\sec^2(y)\,dy = 4(\tan(y))\Big|_0^{\pi/4}$

$= 4(\tan(\frac{\pi}{4}) - \tan(0)) = 4(1-0) = 4$

Sol 2: $\iint_R x\sec^2(y)\,dA = \int_{x=1}^{3}\int_{y=0}^{\pi/4} x\sec^2(y)\,dy\,dx$

inner int: $x\int_0^{\pi/4}\sec^2(y)\,dy = x\tan(y)\Big|_0^{\pi/4} = x(\tan(\frac{\pi}{4}) - \tan(0))$

$= x(1-0) = x$

$\int_1^3 x\,dx = \frac{x^2}{2} = \frac{1}{2}(x^2)\Big|_1^3$

$= \frac{1}{2}(9-1) = 4$

Ex compute $\iint_R \frac{1}{1+x+y}\, dA$ on $R = [1,2] \times [2,3]$

sol: $\iint_R \frac{1}{1+x+y}\, dA = \int_2^3 \int_1^2 \frac{1}{1+x+y}\, dx\, dy$

inner $= \int_1^2 \frac{1}{1+x+y}\, dx$ $\quad u = 1 + x + y$, $du = 1\, dx$

$= \int_1^2 \frac{1}{u}\, du = \ln|u| \Big|_1^2$

$= \ln(1+x+y) \Big|_1^2 = \ln(1+2+y) - \ln(1+1+y)$

$= \ln(|3+y|) - \ln(|2+y|)$

$= \int_2^3 \ln(|3+y|) - \ln(|2+y|)\, dy$ (all possitive so don't necessarily need | | (abs value))

$= 6 \cdot (\ln(6) - 1) - 5(\ln \cdot ($

$= 5(\ln(5) - 1) - 4(\ln(4) - 1))$

$\int \ln(w)\, dw$ $\quad u = \ln(w)$ $\quad dv = dw$

$du = \frac{1}{w} dw$ $\quad v = w$

$w \ln(w) - \int \frac{w}{w}\, dv$

$= 6\ln(6) - 10\ln(5) + 4\ln(4)$ $\quad = w(\ln(w) - 1) + C$

$\cancel{-6 + 5 + 5 - 4}$

$= 6\ln(6) - 10\ln(5) + 4\ln 4$